Image Processing Fundamentals

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Do What? How?

- Find the surface normal to the embedded surface?
- Locate an object within the volume?
- Take the derivative of a kidney?
- How do I evaluate the error introduced by sampling?
- What is aliasing, really?
- How does this all work together in volume graphics?

Image Basics

- Given a discrete volume dataset, \( \text{vol}[x][y][z] \)
- Imagine a volume generating function, \( f \), such that for any particular point \( p_i = (x_i,y_i,z_i) \):
  \[ f(x_i,y_i,z_i) = \text{vol}[x_i][y_i][z_i] \]
- Must reconstruct \( f(x,y,z) \) from \( \text{vol}[x][y][z] \).
- Can make measurements of \( f(x,y,z) \), (e.g., derivatives)
- \textbf{HOW?}

Convolution

- Sampling.
- Interpolation.
- Reconstruction.
- Low pass filtering.
- Noise suppression.
- Edge enhancement.
- Derivative measurement.
- Linear scale space analysis.

Convolution

\[
\text{Convolution integral} \\
\int_{-\infty}^{\infty} h(\tau) I(x - \tau) \, d\tau
\]

Convolution & the Gaussian

\[
\text{Convolution integral} \\
\int_{-\infty}^{\infty} h(\tau) I(x - \tau) \, d\tau
\]

Gaussian as a convolution kernel (with spatial scale parameter, \( \sigma \))

\[
h(x) = G(\sigma, x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}
\]
**Smoothing - noise filtering**

Input, $\phi(x)$ ⊗ Kernel, $G(x, \sigma)$ = $\phi(x) \otimes G(x, \sigma)$

**Linear Scale Space**

\[ \mu(x|\sigma) = G(\sigma, x) \otimes I(x) = \int G(\sigma, t) I(x-t) dt \]

**Properties of the Convolution Operator**

- **Property:** Commutative.
- **Mathematically:** $f \otimes h = h \otimes f$
- **Property:** Associative
- **Mathematically:** $(f \otimes h) \otimes g = f \otimes (h \otimes g)$
- **Property:** Distributive over addition.
- **Mathematically:** $f \otimes (h + g) = (f \otimes h) + (f \otimes g)$

**Convolution & differentiation**

- **Problem:** Real-world data are seldom accompanied with continuous functions, suitable for differentiation.
- **Observation:** Differentiation is just a convolution!
- **Mathematically:** $\frac{\partial}{\partial x} = \frac{\partial}{\partial x} \otimes \phi$
- **Problem:** Derivatives of real-world data are not always defined.
- **Observation:** Differentiation is associative and commutative.
- **Mathematically:** $h(x) \otimes \frac{\partial}{\partial x} \otimes f(x) = \frac{\partial}{\partial x} \otimes h(x) \otimes f(x)$
- **Solution:** Combine differentiation with a regularizing kernel.

**Differentiation**

Not-differentiable

NaN

1/(2\pi)

-20 0 20
-10 0 10
-20 0 20
-10 0 10
-20 0 20
-10 0 10
-20 0 20
-10 0 10
Now for something different...

But not completely different...

\[ e^{i\pi} = -1 \]

\[ e^{\alpha+i\beta} = \cos \alpha + i \sin \beta \]
The Fourier Transform

- A different mirror with which to view images.
- The Fourier transform
  \[ \mathcal{F}(\phi(x)) = \int_{-\infty}^{\infty} \phi(x)e^{-2\pi i \nu x} \, dx = \Phi(\nu) \]
- The result transforms a function of \( x \) in image space to a function of \( \nu \) in frequency space.
- There is an Inverse Fourier transform for undoing this process.
  \[ \mathcal{F}^{-1}(\Phi(\nu)) = \int_{-\infty}^{\infty} \Phi(\nu)e^{2\pi i \nu x} \, d\nu = \phi(x) \]

Fourier Transforms of common functions

<table>
<thead>
<tr>
<th>Image Space:</th>
<th>Frequency Space:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Box (nearest neighbor)</td>
<td>F(Box) sinc</td>
</tr>
<tr>
<td>Pyramid (linear interpolant)</td>
<td>F(Pyramid) (linear interpolant)</td>
</tr>
<tr>
<td>Gaussian (std. distribution)</td>
<td>Gaussian (non-normalized)</td>
</tr>
<tr>
<td>Comb / Shah (sampling function)</td>
<td>Comb</td>
</tr>
</tbody>
</table>

Properties of the Fourier Transform

- Frequency Scaling \( \Rightarrow \) Inverse Spatial Scale Change
  \[ \frac{1}{|a|} \Phi \left( \frac{\nu}{a} \right) = \Phi(\nu a) \]
- Spatial Scaling \( \Rightarrow \) Inverse Frequency Scale Change
  \[ \mathcal{F}^{-1} \left( \frac{1}{|a|} \phi \left( \frac{\nu}{a} \right) \right) = \frac{1}{|a|} \phi(\nu a) \]

The Convolution Theorem

- Convolution in space = multiplication in frequency
  \[ \mathcal{F}(\phi(x) \otimes h(x)) = \Phi(\nu)H(\nu) \]
- Convolution in frequency = multiplication in space
  \[ \mathcal{F}^{-1}(\Phi(\nu) \otimes H(\nu)) = \phi(x)h(x) \]

Revisiting Convolution

now see convolution as a transform, a multiplication, and an inverse transform.
\[ \phi(x) \otimes h(x) = \mathcal{F}^{-1}(\mathcal{F}(\phi(x)) \mathcal{F}(h(x))) \]

Revisiting Convolution (2)
Revisiting Convolution (3)

Input, $\phi(x)$ ⊗ Kernel, $g(x, \sigma) = \phi(x) \otimes g(x, \sigma)$

$F[Input], \Phi(x) \times Kernel, G(x, \sigma) = \Phi(x) \times G(v, \sigma)$

Discrete Fourier Transforms

- Periodic.
  - Assume that the function is a single period of an infinitely repeating function.
  - Or: Think of it as an image that wraps onto itself like a doughnut (torus).
- Discrete.
  - If there are $n$ samples in the spatial domain, there will be $n$ samples in frequency domain, too.

Discrete Fourier Transforms (1)

Discrete Fourier Transforms (2)

Discrete Fourier Transforms (3)

Sampling

now see sampling as a multiplication with frequency issues.
Note: adding more (higher) frequencies alters the period of the ringing, but does not reduce the amplitude.

A signal \( \phi \) is considered bandlimited if its Fourier transform, \( \Phi(\omega) = \Phi(-\omega) = 0 \) for all \( \omega > \omega_{\text{limit}} \). Satisfies the Nyquist criterion.

The discrete signal does not contain frequencies higher than 1/2 the sampling frequency.

Signal that isn’t Bandlimited.
Sampling effects.

Source of Aliasing

Non-bandlimited signal
Low sampling rate (below Nyquist)
Non perfect reconstruction

Possible Errors

Post-aliasing
Reconstruction (filter passes frequencies beyond the Nyquist frequency (of duplicated frequency spectrum)) \( \Rightarrow \) frequency components of the original signal appear in the reconstructed signal at different frequencies.
Smoothing
Frequency below the Nyquist frequency are attenuated.

Possible Errors (2)

Ringing (overshoot)
Occurs when trying to sample/reconstruct discontinuity.
Anisotropy
Caused by non-spherically symmetric filters.
Requires filters that are invariant with respect to rotation.
input

- Bandlimited.
- Appropriately sampled: above the Nyquist frequency.

Reconstruction - Interpolation

Example in 1D

Interpolation (summary)

- Very important; regardless of algorithm
  - expensive \( => \) done very often for one image
- Requirements for good reconstruction
  - performance
  - stability of the numerical algorithm
  - accuracy

Nearest neighbor

Linear
the pipeline (westover ’91)

Transformation magnify / minify

- Remember, spatial scaling = inverse frequency scale.
- Magnification / scaling of the reconstructed input ⇒ transformation / minimization of the sampling function.
- Minification / scaling of the reconstructed input ⇒ transformation / magnification of the sampling function.

Resample

- with antialiasing
- without antialiasing

Summary

For (n = 1; n < image_size; n++)
\[
\text{gradx}[n] = \text{image}[n-1] \times -0.5 + \text{image}[n+1] \times 0.5;
\]

You’re really approximating...

With all the consequences in the frequency domain!

Summary

- Convolution is a basic operation, used in:
  - Interpolation, reconstruction.
  - Noise filtering.
  - Differentiation, measurement.
  - Statistics.
- The frequency domain.
  - It exists.
  - Operations in discrete images have frequency based consequences.
  - It’s happening whether you’re watching for it or not.